Lattices and their theta functions

Lattices

Definition: A *lattice* in $\mathbb{R}^d$ is a discrete abelian subgroup of full rank d.

Examples:

$\mathbb{Z} \subset \mathbb{R}$ is a lattice ✓

$\mathbb{Z} + \sqrt{2}\cdot\mathbb{Z} \subset \mathbb{R}$ is not a lattice ✗

$\mathbb{Z} \times \{0\} \subset \mathbb{R}^2$ is not a lattice ✗

Notation: For $x, y \in \mathbb{R}^d$ we denote by (x,y) the standard Euclidean product $(x,y) = x_1y_1 + x_2y_2 + \dots + x_dy_d$

$|x| = (x_1^2 + \dots + x_d^2)^{1/2}$ - Euclidean length

Definition: Let $\Lambda \subset \mathbb{R}^d$ be a lattice and let $\mathcal{B} = \{b_1, \dots, b_d\}$ be a $\mathbb{Z}$-basis of Λ.

A Gram matrix of Λ associated to $\mathcal{B}$ is

$$G := \left((b_i, b_j)\right)_{i,j=1}^{d}.$$

symmetric & positive definite matrix

Any two Gram matrices, say G and G', of Λ are related by:

$$G = \mathcal{U} G' \mathcal{U}^t$$

for some $\mathcal{U} \in GL_d(\mathbb{Z})$

Definition: Let $\Lambda \subset \mathbb{R}^d$ be a lattice
its dual lattice

$$\Lambda^* := \{ m \in \mathbb{R}^d \mid (m, \ell) \in \mathbb{Z} \text{ for all } \ell \in \Lambda \}$$

Example: $\Lambda = a \cdot \mathbb{Z}$ for some $a \in \mathbb{R}$.
Then $\Lambda^* = \frac{1}{a} \cdot \mathbb{Z}$

Exercise:
If $\Lambda = B \cdot \mathbb{Z}^d$ for some matrix $B \in GL_n(\mathbb{R})$
Then $\Lambda^* = B^{-t} \cdot \mathbb{Z}^d$ Notation: $B^{-t} = (B^{-1})^t = (B^t)^{-1}$

Definition: A lattice $\Lambda \subset \mathbb{R}^d$ is

- **integral** if $(\ell_1, \ell_2) \in \mathbb{Z}$ for all $\ell_1, \ell_2 \in \Lambda$

Λ is integral $\Leftrightarrow$ Gram matrix of Λ has all integer elements
Λ is integral $\Leftrightarrow$ $\Lambda \subset \Lambda^*$

- **even** if $|\ell|^2 = (\ell, \ell) \in 2\mathbb{Z}$ for all $\ell \in \Lambda$

Exercise: Λ is even $\Leftrightarrow$ Gram matrix of Λ has all integer elements and all its diagonal elements are even.
We call such matrices **even**.

Example: lattice $\mathbb{Z}$ is integral and not even.
lattice $2 \cdot \mathbb{Z}$ is integral and even

- **unimodular** if $\mathrm{vol}(\mathbb{R}^d / \Lambda) = 1$

Exercise: Let G be a Gram matrix of Λ. Then $\mathrm{vol}(\mathbb{R}^d/\Lambda) = \det(G)^{1/2}$.

Definition: Let Λ be an even lattice. The **level** of Λ is the minimal positive integer N such that the lattice $N^{1/2} \cdot \Lambda^*$ is even.

Let G be a Gram matrix of Λ. G is an even matrix, and N is the smallest positive integer such that $N \cdot G^{-1}$ is an even matrix.

Note that G^{-1} has all rational elements $\Rightarrow$ $N \cdot G^{-1}$ is even for some $N \in \mathbb{Z}$.

Definition: The *theta function* of a lattice $\Lambda \subset \mathbb{R}^d$ is

$$\Theta_\Lambda(z) = \sum_{\ell \in \Lambda} e^{\pi i |\ell|^2 z}, \quad z \in \mathfrak{h}.$$

The goal of our lecture is to investigate modular properties of Θ_Λ.

Theorem: Let $\Lambda \subset \mathbb{R}^n$ be an even lattice of level N. Let G be a Gram matrix of Λ. Then for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$

$$\Theta_\Lambda\left(\frac{az+b}{cz+d}\right) = \chi_G\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \sqrt{cz+d}^{\,n} \cdot \Theta_\Lambda(z)$$

$$\chi_G\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{\sqrt{\frac{-1/\tau - c/d}{i}} \cdot \sqrt{\frac{\tau}{i}}}{\sqrt{c\tau+d}} \right)^n \cdot \sum_{p \in \mathbb{Z}^n / d\mathbb{Z}^n} e^{\pi i b (p, Sp)/d}$$

Recall: $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \mod N \right\}$

Lemma:

If n is even then $\chi_G : \Gamma_0(N) \to \{\pm 1\}$ is a group character and

$$\chi_G \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \left(\frac{d}{|d|}\right)^{n/2} & \text{for } c = 0 \\ \left(\frac{(-1)^{n/2} \cdot \det G}{p}\right) & \text{for } c \neq 0 \end{cases}$$

Legendre symbol

in the case p is an odd prime number of the form $p \equiv d \pmod{c}$.

We will prove this result during our next lectures

Definition: Let $f : \mathbb{R}^d \to \mathbb{C}$ be an L^1 function. The Fourier transform of f is

$$\hat{f}(y) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i\, x \cdot y}\, dy$$

Poisson summation formula: Let $\Lambda \subset \mathbb{R}^d$ be a lattice. Then

$$\sum_{\ell \in \Lambda} f(\ell) = \frac{1}{\operatorname{vol}(\mathbb{R}^d/\Lambda)} \cdot \sum_{m \in \Lambda^*} \hat{f}(m)$$

Lemma: Let $\Lambda \subset \mathbb{R}^d$ be a lattice. Then

$$(-iz)^{-d/2}\, \Theta_\Lambda\left(-\frac{1}{z}\right) = \frac{1}{\operatorname{vol}(\mathbb{R}^d/\Lambda)}\, \Theta_{\Lambda^*}(z)$$

Lemma: Let Λ be an even lattice. Then

$$\Theta_\Lambda(z+1) = \Theta_\Lambda(z)$$

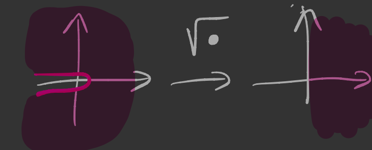
Exercise: even + unimodular $\Downarrow$ $\Lambda^* = \Lambda$

Proof: (very easy) Exercise

The easiest transformation law: Λ is even and unimodular.

In this case

Branch of $\sqrt{\cdot}$



$$\Theta_\Lambda\left(-\frac{1}{z}\right) = (-iz)^{d/2}\, \Theta_\Lambda(z)$$

$$\Theta_\Lambda(z+1) = \Theta_\Lambda(z)$$

Theorem: Let $\Lambda \subset \mathbb{R}^d$ be an even unimodular lattice. Then d is divisible by 8 and $\Theta_\Lambda \in M_{d/2}(\Gamma_1)$.

It suffices to show that d can not be an odd multiple of 4. (☆)

d odd $\Rightarrow$ $\Lambda \oplus \Lambda \oplus \Lambda \oplus \Lambda$ is an even unimodular lattice in $\mathbb{R}^{4d}$ $\Rightarrow$ Λ can not exist by (☆)

d is an odd multiple of 2 $\Rightarrow$ $\Lambda \oplus \Lambda$ can not exist by (☆)

Lemma: Suppose that f is a cusp form of weight $k \in 2\mathbb{Z}_{>0}$ and group Γ_1, and f has the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} c_f(n) e^{2\pi i n z}$$

Then $c_f(n) = O(n^{k/2})$ as $n \to \infty$

Proof: Let $f \in S_k(\Gamma_1)$.

Then the function $|f(z)|\,\mathrm{Im}(z)^{k/2} =: g(z)$ is Γ_1-invariant.

Moreover, $g(z)$ is bounded as $z \in \mathfrak{h}$

Suppose $|g(z)| < C$, $z \in \mathfrak{h}$ for some $C \in \mathbb{R}_{>0}$

We estimate the Fourier coefficients of f in the following way:

$y_0 > 0$

$$C_f(n) = \int_{iy_0}^{iy_0+1} f(z) e^{-2\pi i n z} \, dz$$

$$|C_f(n)| = \left| \int_{iy_0}^{iy_0+1} f(z) e^{-2\pi i n z} \, dz \right| = \left\{ \begin{array}{c} \text{we choose} \\ y_0 = \frac{1}{n} \end{array} \right\} = \left| \int_{i/n}^{i/n+1} f(z)\,\mathrm{Im}(z)^{k/2} \cdot \mathrm{Im}(z)^{-k/2} e^{-2\pi i n z} \, dz \right|$$

$$\leq \int_{i/n}^{i/n+1} \left| f(z)\,\mathrm{Im}(z)^{k/2} \right| \cdot n^{k/2} \cdot e^{2\pi} \cdot \left| e^{-2\pi i n \cdot \mathrm{Re}(z)} \right| dz \leq C \cdot e^{2\pi} \cdot n^{k/2}$$

This finishes the proof of the lemma $\square$

Since $M_k(\Gamma_1) = \mathbb{C} \cdot E_k \oplus S_k(\Gamma_1)$, the theta function can be written as

$$\Theta_\Lambda(z) = E_{d/2}(z) + f(z) \text{ for some } f \in S_{d/2}(\Gamma_1)$$

We know that the Fourier expansion of E_k is

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z}$$

B_k is negative if k is divisible by 4 and positive if k is an odd multiple of 2.

Fourier coefficients of E_k grow at least like n^{k-1}

If d is an odd multiple of 4, then some coefficients of Θ_Λ are negative $\lightning$

This finishes the proof of the theorem $\blacksquare$

The smallest dimension where an even, unimodular can exist is $d = 8$.

Such a lattice exists, it is the E_8-lattice

$$\Lambda_{E_8} := \left\{ (x_i) \in \mathbb{Z}^8 \cup \left(\mathbb{Z} + \tfrac{1}{2}\right)^8 \;\middle|\; \sum_i x_i \equiv 0 \pmod 2 \right\}$$

Exercise:
- Show that Λ is a lattice
- Show that Λ is even and unimodular

Lemma:

$$\Theta_{\Lambda_{E_8}}(z) = E_4(z).$$